Journal of Statistical Physics, Vol. 38, Nos. 1/2, 1985

Exact Solution of the Multichannel Kondo Problem, Scaling, and Integrability

A. M. Tsvelick[1] **and P. B. Wiegmann**[1]

Received April 2, 1984

In this work we give the exact solution of the model describing the scattering of conduction electrons by an impurity in the orbital singlet state (so-called n-channel Kondo problem). Depending on the relation between the impurity spin S and the number of electron scattering channels n, the model behaves differently at low energies. At $n \leqslant 2S$ the effective charge increases to infinity at low energies, whereas at $n > 2S$ it tends to a finite fixed point. The model under study is the first example of the one-dimensional quantum field theory exhibiting scaling behavior.

KEY WORDS: Kondo problem; dilute magnetic alloys; renormalization group; scaling; fixed point; Bethe Ansatz.

1. INTRODUCTION

Recently it has been shown that many traditional models which are conventionally used to investigate properties of dilute magnetic alloys are completely integrable and their exact solution has been obtained on the basis of the Bethe–Ansatz technique (see review, Refs. 1–3). The intriguing exception was the so-called multichannel Kondo model the exact solution of which has not been so far derived although all experts agree that it exists.

The Hamiltonian of the multichannel Kondo model is

$$\mathscr{H} = \sum_{k,m,\sigma} (k - k_F)\, C^+_{km\sigma} C_{km\sigma} + J \sum_{\substack{k,k' \\ m,\sigma,\sigma'}} C^+_{km\sigma} \boldsymbol{\sigma}_{\sigma\sigma'} C_{k'm\sigma'} \mathbf{S} \qquad (1.1)$$

We suppose that the electron plane wave are expanded in spherical waves with a center placed in the impurity point. Impurity interacts only with that

[1] L. D. Landau Institute for Theoretical Physics, USSR Academy of Sciences, 117334 Moscow, ul.Kosygina 2, USSR.

0022-4715/85/0100-0125$04.50/0

822/38/1-2-9

partial wave which have $l = l_0$ (l_0 is an unfilled impurity shell). The state of the partial wave is characterized by a modulus of momentum k, projection of an angular momentum m ($m = -l_0, ..., l_0$), and spin σ. $C^+_{km\sigma}$ is a creation operator of this state.

The direct application of the Bethe method to this model meets with a difficulty which is accounted for by the incorrect treatment of high energy processes in the framework of the naive Bethe-Ansatz scheme.

The multichannel exchange Hamiltonian seems to be the most interesting among the Kondo models both from the "magnetic" and field theory points of view. First at $n = 2S$ the Hamiltonian (1.1) is realistic and describes three-dimensional impurities with half-filled atomic shell. In this case Hund's rule asserts that the ion ground state is an orbital singlet with the spin $S = l + 1/2$ (therefore the model is often reffered to as the "Kondo problem for the orbital singlet.") This is the case with Mn which has $n = 5$ d electrons and the configuration ${}^6S_{5/2}$. The scattering therefore involves only exchange of spins, whereas the projection of the orbital angular momentum of the l-electron wave remains unchanged.[4]

An orbital singlet may also appear as the result of the combined action of Hund's rule and the crystal field. This holds for V ($n = 3$) and co-impurities ($n = 8$) in a cubic crystal field.[5] At $n \neq 2S$ the Hamiltonian (1.1) can serve as a model except the exotic case of alloys with extremely low Kondo temperatures when the Kondo temperature is compared with hyperfine coupling.

Second the model with $n > 2S$ serves as a unique example of the quantum field theory which is asymptotically free and has a finite infrared fixed point.

In all known asymptotically free quantum one-dimensional field theories the effective interaction increases rapidly with a decrease of the energy scale until the unitary limit is achieved. Such a dramatic role of the interaction in the formation of the ground state of the system is primarely due to one dimensionality and the common lore is that in such systems the fixed point can be either zero or infinite.

Recently, Nozieres and Blandin[5] in their work have pointed out that the multichannel Kondo Hamiltonian (1.1) in the case $n > 2S$ is the first exception from the rule.[2]

They have given simple plausible arguments that the fixed point of the Hamiltonian (1) is finite at $n > 2S$ and corresponds to some finite exchange interaction J^*. As a consequence all physical quantities should obey the scaling power low at small energy scale. For all $n \leqslant 2S$ the fixed point

[2] The second exception is the Wess–Zumino chiral field theory,[6] which is close to the model under study.

corresponds to the strong coupling limit. Here we give the Bethe-Ansatz solution of the model for arbitrary n and S and investigate qualitatively three different cases: $n = 2S$, $n < 2S$, and $n > 2S$. The results of the present work have been briefly published in Ref. 7.

2. NOZIERES AND BLANDIN ARGUMENTS AND COMPARISON WITH THE EXACT SOLUTION

It is well known that the exchange Hamiltonian (1.1) is renormalizable (see, for instance, Refs. 8 and 9). It means that all physical quantities depend on the energy scale ε, Fermi energy ε_F, and bare coupling constant J only through dimensionless value ε/T_K, where T_K is the Kondo temperature dependent on J and ε_F. The conventional method of describing renormalization properties of the theory is to consider the effective-dependent exchange amplitude $z(\varepsilon)$ satisfying the Gell–Mann–Low equations

$$\frac{dz(\varepsilon)}{d\ln\varepsilon} = \beta(z, n), \qquad z(\varepsilon_F) = J \tag{2.1}$$

Then physical quantity (for instance, impurity magnetization $M_{\text{imp}}(H/T_K)$ satisfies the equation

$$\frac{d\ln M}{d\ln H} = f(z(H)), \qquad M(\varepsilon_F) = S \tag{2.2}$$

The function $\beta(z)$ and $f(z)$ can be treated by perturbation theory, the two-loops contribution of which for the model (1.1) is[5]

$$\beta(z) = -z^2 + nz^3 + O(z^4), \qquad f(z) = -z^2 + O(z^4) \tag{2.3}$$

It means that

$$M_{\text{imp}}(H/T_K) = S - \sum_{n=1}^{\infty} a_n z_0^n(H/T_K) \tag{2.4}$$

where

$$z_0^{-1} - n\ln z_0 = \ln H/T_K, \qquad z_0(\varepsilon_F) = J \tag{2.5}$$

and $a_1 = S$. In particular, it follows from these equations that the weak coupling limit is unstable and is a repulsive fixed point at the antiferromagnetic exchange. The effective charge increases with decreasing energy scale. And vice versa at the ferromagnetic interaction the weak coupling limit is stable and is an attractive fixed point.

What is the value of the charge in the stable (repulsive) fixed point in the case of the antiferromagnetic exchange?

At $n = 1$ numerous researches testify to the fact that is the strong coupling limit.

This fact is also established on the basis of the exact solution of the $n = 1$ problem.[10] There is no doubt that the strong coupling limit is the stable fixed point for $n \leqslant 2S$.

Nozieres and Blandin in their work have given convincing qualitative arguments that it is not true for $n > 2S$. Let us repeat briefly their arguments.

First consider the case $n < 2S$. Clearly, in the string coupling limit the impurity traps as many electrons as it is permitted by the Pauli principle, i.e., one electron with a spin opposite to the impurity spin per an orbital channel.

The electrons cannot quench the impurity spin completely and it remains magnetic with the spin $S - n/2$ (see Fig. 1a). This complex can interact with other electrons and generate a new Kondo effect. But the new exchange amplitude is ferromagnetic in that only electrons with the spin antiparallel to the bounded electrons can interact with the impurity complex and decrease the energy. But their spins are parallel to the spin of the impurity complex and therefore the new exchange is ferromagnetic. Therefore the strong coupling limit is a stable fixed point. It means that at $H \to 0$ the magnetization $M_{\text{imp}}(H) \to S - n/2$, and the approach to this fixed point is governed by the logarithmic law due to the new Kondo effect:

$$M_{\text{imp}}(H) = (S - n/2)\left(1 - \frac{1}{\ln H/T_K} + \cdots\right) \qquad \text{at} \quad H \ll T_K \qquad (2.6)$$

At $n = 2S$ the low-energy fixed point corresponds also to the strong coupling limit. The ground state is nondegenerate and is formed by a singlet

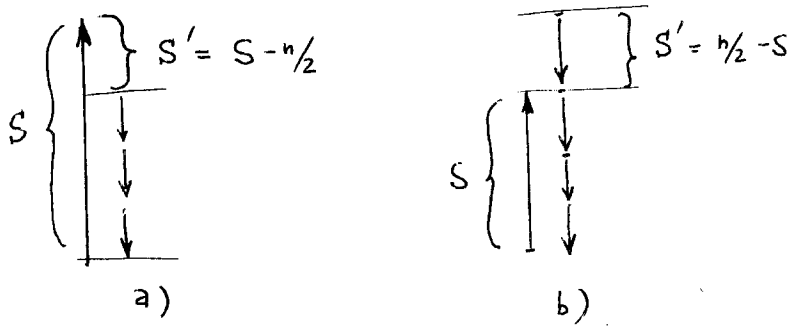

Fig. 1

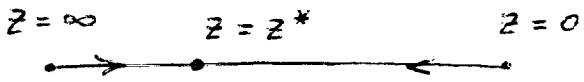

Fig. 2

combination of one impurity electron and one conduction electron in each orbital channel. In this situation the Nozieres Fermi liquid picture is applied.[11] The polarizability of the ground state is finite at $H \to 0$:

$$M_{\text{imp}}(H) = H/2\pi T_K + O(H^3/T_K^3) \tag{2.7}$$

At $2S < n$ again an electron with the spin opposite to the spin of the bounded electrons can interact with the impurity complex but now this interaction is antiferromagnetic (see Fig. 1b). Therefore the strong coupling limit is unstable. The scaling trajectory are then locked between the two unstable points and must converge to a finite fixed point. It means that the Gell-Mann–Low function has a zero $\beta(z^*) = 0$ (Fig. 3), and

$$\beta(z) = \beta_0 \cdot (z - z^*), \qquad f(z) = \alpha_0 \cdot (z - z^*) \tag{2.8}$$

As a conclusion the physical quantities have power law scaling behavior at the small energy scale. For example,

$$M_{\text{imp}}(H) = (H/T_K)^{\beta_0} \tag{2.9}$$

where β_0 is a number dependent only on n and $2S$.

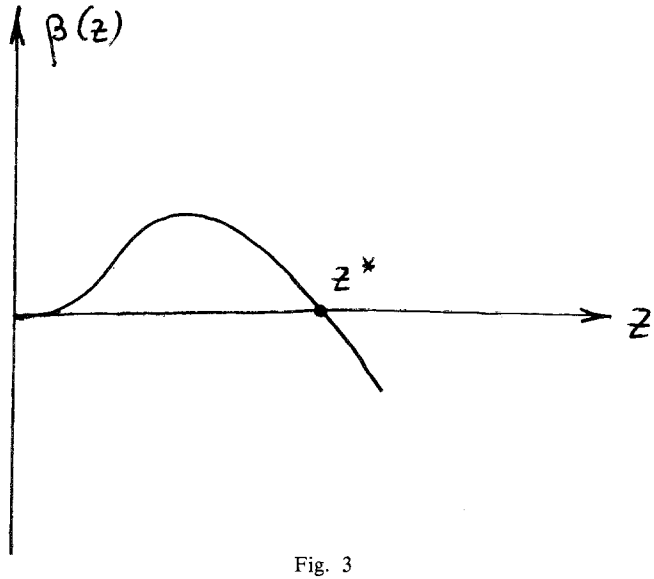

Fig. 3

Below on the basis of the exact solution we obtain the explicit expression for the impurity magnetization:

$$M_{\text{imp}}(H) = (S - n/2)\,\theta(S - n/2) - \frac{in}{4\pi^{3/2}} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega - i0}$$
$$\times \exp[2i\omega \ln(H/T_H)] \times \left(\frac{i\omega n + 0}{e}\right)^{i\omega n}$$
$$\times \frac{\Gamma(1 + i\omega)\,\Gamma(\frac{1}{2} - i\omega)}{\Gamma(1 + i\omega n)} [\exp(-\pi\,|n - 2S|\,|\omega|)] \quad (2.10)$$
$$- \exp[-\pi(n + 2S)\,|\omega|)][1 - \exp(-2\pi n\,|\omega|)]^{-1}$$

$$T_H = \frac{2\pi(n/2e)^{n/2}}{\Gamma(n/2)} T_K, \qquad \begin{array}{l} T_K \text{ is the Kondo temperature,} \\ \theta \text{ is a } \theta \text{ function} \end{array}$$

Compare this expression with the perturbation theory and Nozieres–Blandin prediction. First, let us consider the high magnetic fields $H \gg T_K$ where the perturbation theory is valid. Introducing $z_0(H)$ through Eq. (2.5) and deforming the integration path to the upper half-plane to envelop the cut of $(i\omega n + 0)^{i\omega n}$ we obtain

$$M_{\text{imp}} = S - \frac{1}{\pi^{3/2}} \int_0^{\infty} \frac{dt}{t} e^{-2t} \frac{\sin \pi n z_0 t \sin 2\pi S z_0 t}{\sin \pi z_0 t}$$
$$\times \frac{\Gamma(1 + nz_0 t)\,\Gamma(\frac{1}{2} + z_0 t)}{\Gamma(1 + z_0 t)} \left(\frac{n}{e}\right)^{-nz_0 t}$$
$$= S - \sum_{k=1}^{\infty} z_0^k (H/T_K)\, a_K(S, n) \quad (2.11)$$

The strong dependence of M_{imp} on n and $2S$ arises in the low-energy regime ($H \ll T_k$). The character of the singularities of the integrand in the lower half-plane depends on n and $2S$. At $n < 2S$ there is a cut and we obtain

$$M_{\text{imp}} = S - n/2 + \sum_{k=1}^{\infty} a_k(S - n/2, n)\, z_0^k(H/T_K) + O(H/T_K) \quad (2.12)$$

At $n = 2S$ there is only poles in integer point the lower half-plane. Then we have

$$M_{\text{imp}} = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \left(\frac{H}{T_H}\right)^{2k+1} \cdot \frac{\Gamma(k + \frac{1}{2})}{\Gamma(n(k + \frac{1}{2}))} \cdot \frac{\{n[(k + \frac{1}{2})/e]\}^{n(k+1/2)}(-1)^k}{k!\,(2k + 1)} \quad (2.13)$$

Finally at $n > 2S$ poles in noninteger points appear:

$$M_{\text{imp}} = \sum_{k=1}^{\infty} b_k \left(\frac{H}{T_H}\right)^{2k/n} + \sum_{k=0}^{\infty} c_k \left(\frac{H}{T_H}\right)^{2k+1} \tag{2.14}$$

3. THE ANDERSON MODEL AS A REGULARIZED VERSION OF THE EXCHANGE HAMILTONIAN

As we have claimed in the Introduction the attempts to diagonalize the Hamiltonian straightforwardly by the Bethe-Ansatz technique encounter a difficulty. To demonstrate it let us recall that two physically clear approximations are necessary for an exchange model to be integrable.[1-3] These are (i) the linear spectrum approximation, (ii) the pointlike interaction approximation.

The both approximations are no doubt valid for physical particles. However in the Bethe-Ansatz approach one should start from the bare vacuum (unfilled Dirac sea) and deal with "bare" particles. Sometimes application of both approximations to bare particles is not so bad and allows to obtain correct universal properties of the system.[1-3]

Here it leads to physically meaningless results and makes it impossible to correctly take into account high-energy processes. More exactly, this approach makes it impossible to take into account to interpret correctly the axial anomaly.

In fact the scattering matrix of a bare Dirac electron by the pointlike impurity does not depend on the energy of the particle. In our case due to the $SU(2) \otimes SU(n)$ symmetry of the exchange it is the tensor product of the matrices acting on the spin and orbital spaces:

$$S_{m\sigma;s}^{m'\sigma';s'} = [\exp(iJ\boldsymbol{\sigma} \cdot \mathbf{S})]_{\sigma s}^{\sigma' s'} \delta_{mm'}$$
$$\left(m, m' = -\left(\frac{n-1}{2}\right),\ldots, \left(\frac{n-1}{2}\right); \sigma, \sigma' = \pm\frac{1}{2}, s, s' = -S,\ldots, +S\right) \tag{3.1}$$

As a result of the particle–particle factorization multiparticle processes in spin and orbital channels seem to be independent.[12] To make sure that this result is invalid, suffice it to consider the two-loop diagram in the standard perturbation theory where the spin and orbital processes are coupled [Eq. (2.5)].

Here we give up the pointlike approximation and consider the orbital degenerate Anderson model of a special type which serves us as a regularizer.

The Hamiltonian (1.1) is the version of the following Anderson model:

$$\mathscr{H} = \sum_{k,m,\sigma} v_F(k-k_F)\, C^+_{km\sigma} C_{km\sigma} + V \sum_{k,m,\sigma} (C^+_{km\sigma} d_{m\sigma} + \text{H.c.}) + \mathscr{H}_{\text{atom}} \tag{3.2}$$

$$\mathscr{H}_{\text{atom}} = E_d \sum_{m,\sigma} d^+_{m\sigma} d_{m\sigma} - \frac{U}{2} \sum_{\substack{m,m' \\ \sigma,\sigma'}} d^+_{m\sigma} d^+_{m'\sigma'} d_{m'\sigma} d_{m\sigma'} \tag{3.3}$$

In the absence of the hybridization ($V=0$), the ground state of the atomic shell is at $0 < U(n-1)/2 < E_d < U(n+2)/2$ a orbital singlet ($n_d = n$, $S = n/2$, $L=0$). For the sake of simplicity we deal only with the asymmetric case and consider the hybridization only of the states $n_d = n$ and $n_d = n-1$, i.e., the parameters satisfy the condition

$$U + \varepsilon_d/2 \gg \Gamma$$

where $-\varepsilon_d = 2E_d - U(n-1)$, $\Gamma = \pi\rho(\varepsilon_F)\, V^2$. The models (1.1) and (3.2), (3.3) become equivalent under the condition

$$-\varepsilon_d \gg n\Gamma \tag{3.4}$$

Actually the states which are not an orbital singlet should be considered only as virtual and then the Schrieffer–Wolf transformation leads us to the exchange Hamiltonian (1.1) at $n = 2S$.

The application of Bethe-Ansatz technique to the Anderson model (3.2), (3.3) does not encounter difficulties. Below we shall give the solution of this model and perform the limit (3.4).

4. BETHE-ANSATZ SOLUTION OF THE ANDERSON MODEL

One can obtain the solution of the model (3.2), (3.3) by glueing together the Bethe-Ansatz solutions for the $SU(2)$- and $SU(n)$-invariant Anderson models. These models are described in details in Refs. 1 and 14 (see also Ref. 3). The case in that due to the $SU(2) \otimes SU(n)$ symmetry of the Hamiltonian (3.2), (3.3) the particle–particle S matrix is also tensor production:

$$S(k-p) = S^\sigma(k-p) \otimes S^m(k-p) \tag{4.1}$$

but it now depends on the energy of the particles. Here S^σ is the S matrix of the nondegenerate Anderson model with repulsion on the atomic shell, it acts

on the spin space; S^m is the S matrix of the degenerate Anderson model with attraction, it acts on the orbital space[13,14]

$$S^\sigma(k) = (k + 2\Gamma i P_\sigma)/(k + 2i\Gamma) \tag{4.2a}$$

$$S^m(k) = (k - 2i\Gamma P_m)/(k - 2i\Gamma) \tag{4.2b}$$

where P_σ, P_m are permutation operators.

The electron scattering phase by the impurity is

$$\phi(k) = 2\tan^{-1}\left(\frac{k - \varepsilon_d}{\Gamma}\right) \tag{4.3}$$

Imposition of the periodic boundary conditions to the Bethe wave function leads to the eigenvalue problem:

$$\exp[ik_j L + i\phi(k_j)] = S_{jj+1}(k_j - k_{j+1}) \cdots S_{jN} S_{j1} \cdots S_{jj-1}(k_j - k_{j-1})$$

solution of which can be found when the spin[13] and orbital[14] Bethe Ansätze are glued together:

$$\exp(2i\Gamma\lambda_j^{(0)} L)\, e_1(\lambda_j^{(0)} - \varepsilon_d/2\Gamma) = t^\sigma(\lambda_j^{(0)})\, t^m(\lambda_j^{(0)}) \tag{4.5}$$

where $t^{\sigma,m}$ are the eigenvalues of the operators

$$T^a(\lambda) = \prod_{\substack{p=1 \\ p\neq j}}^{N} S^a_{jp}(\lambda - \lambda_p^{(0)}), \qquad (a = \sigma, m)$$
$$e_n(x) = (x - in/2)/(x + in/2) \tag{4.6}$$

The eigenvalues of the Hamiltonian (6) are

$$E = 2\Gamma \sum_{j=1}^{N} \lambda_j^{(0)} \tag{4.7}$$

and they do not split into independent spin and orbital parts. They are coupled by Eq. (4.5) owing to the energy dependence of the bare S matrix. The eigenvalues $t^{\sigma,m}$ are determined by the Bethe-Ansatz hierarchies:

$$t^\sigma(\lambda_j^{(0)}) = \prod_{\alpha=1}^{M} e_1(\lambda_j^{(0)} - \Lambda_\alpha) \tag{4.8a}$$

$$\prod_{j=1}^{N} e_1(\Lambda_\alpha - \lambda_j^{(0)}) = \prod_{\beta=1}^{M} e_2(\Lambda_\alpha - \Lambda_\beta) \tag{4.8b}$$

$$t^m(\lambda_j^{(0)}) = \prod_{\beta=1}^{m^{(1)}} e_1(\mu_\beta^{(1)} - \lambda_j^{(0)}) \tag{4.8c}$$

$$\prod_{\tau=\pm 1} \prod_{\beta=1}^{m^{(j+\tau)}} e_1(\mu_\alpha^{(j)} - \mu_\beta^{(j+\tau)}) = \prod_{\beta=1}^{m^{(j)}} e_2(\mu_\alpha^{(j)} - \mu_\beta^{(j)}) \tag{4.8d}$$

where $m^{(0)} \equiv N-$ is the total number of particles, $M = N/2 - S^z$ is the number of particles with up spin,

$$m^{(j)} = \sum_{k=j+1}^{n} n_k \tag{4.9}$$

where n_k is the number of particles with the $[(n+1)/2 - k]$th projection of the angular momentum. The total spin and orbital projection are

$$S^z = N/2 - M, \qquad L^z = \frac{(n-1)N}{2} - \sum_{j=1}^{n-1} m_j \tag{4.10}$$

5. SPECTRAL EQUATIONS

The gluing of the two hierarchies (4.8) brings out considerable sophistication of the ground state.

At first let us give the classification of the solution of Eqs. (4.8) in the thermodynamic limit.

(a) The pairs of rapidities $\lambda^{(0)}$s can form the bound state with some Λs:

$$\lambda^{(0)} = \Lambda \pm i/2 \tag{5.1}$$

Let $\sigma(\Lambda)$ and $\tilde{\sigma}(\Lambda)$ be distributions of these Λs and their holes.

(b) Let $\rho(\lambda)$ and $\tilde{\rho}(\lambda)$ be distributions of the unbounded real λs and their holes.

(c) The others Λs and the orbital rapidities $\mu^{(j)}$s can form the complexes of the order $m = 1, 2, \ldots, \infty$:

$$\Lambda^{(m,k)} = \Lambda^{(m)} + i\left(\frac{m+1}{2} - k\right) \qquad (k = 1, \ldots, m) \tag{5.2a}$$

$$\mu_j^{(m,k)} = \mu_m^{(j)} + i\left(\frac{m+1}{2} - k\right) \qquad (k = 1, \ldots, m) \tag{5.2b}$$

Let σ_m, $\tilde{\sigma}_m$ and $\kappa_m^{(j)}$, $\tilde{\kappa}_m^{(j)}$ be distributions of centers of the complexes and their holes.

Performing the necessary manipulations we obtain the spectral equations for a set of distributions[3,15] (for a general techniqualities see Ref. 16).

$$\frac{\Gamma}{\pi}+\frac{1}{L}a_1(\lambda-\varepsilon_d/2\Gamma)=\tilde{\rho}(\lambda)+\rho(\lambda)+a_1*\sigma(\lambda)+a_m*\sigma_m(\lambda)$$
$$-s*A_{1m}*\kappa_m^{(1)} \tag{5.3a}$$

$$\frac{2\Gamma}{\pi}+\frac{1}{L}a_2(\lambda-\varepsilon_d/2\Gamma)=\tilde{\sigma}(\lambda)+(1+a_2)*\sigma(\lambda)+a_1*\rho(\lambda)$$
$$-s*A_{2m}*\kappa_m^{(1)}(\lambda) \tag{5.3b}$$

$$(a_l*\rho+A_{2l}*S*\sigma)\,\delta_{j1}=\tilde{\kappa}_l^{(j)}+A_{lp}*C^{jk}*\kappa_p^{(k)} \tag{5.3c}$$

$$a_l*\rho=\tilde{\sigma}_l+A_{lp}*\sigma_p \tag{5.3d}$$

The energy, spin, and orbital moment of the state described by the set of distributions (5.3) are

$$\frac{E}{L}=2\Gamma\int_{-\mathscr{D}^{(-)}/2\Gamma}^{\mathscr{D}^{(+)}/2\Gamma}\lambda(2\sigma(\lambda)+\rho(\lambda))\,d\lambda \tag{5.4a}$$

$$\frac{N}{L}=2\int_{-\mathscr{D}^{(-)}/2\Gamma}^{\mathscr{D}^{(+)}/2\Gamma}\sigma(\lambda)\,d\lambda+\int\rho(\lambda)\,d\lambda \tag{5.4b}$$

$$\frac{S^z}{L}=\frac{1}{2}\int\rho(\lambda)\,d\lambda-\sum_{m=1}^{\infty}m\int\sigma_m(\lambda)\,d\lambda \tag{5.4c}$$

$$\frac{L^z}{L}=\left(\frac{n-1}{2}\right)\frac{N}{L}-\sum_{m=1}^{\infty}\sum_{j=1}^{n-1}m\int\kappa_m^{(j)}(\lambda)\,d\lambda \tag{5.4d}$$

where $\mathscr{D}^{(\pm)}$ are the edges of the conductive band. Here we used the convolution symbol

$$f*g(\lambda)=\int_{-\infty}^{+\infty}f(\lambda-\lambda')\,g(\lambda')\,d\lambda'$$

and notations

$$a_m(\lambda)=\frac{2}{\pi}\frac{m}{m^2+4\lambda^2} \tag{5.5a}$$

$$s(\lambda)=(2\cosh\pi\lambda)^{-1} \tag{5.5b}$$

$$C^{jk}(\lambda)=\delta_{jk}\delta(\lambda)-s(\lambda)(\delta_{j,k+1}+\delta_{j,k-1}) \tag{5.5c}$$

$$A_{nm}(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\,e^{-i\omega\lambda}\coth\frac{|\omega|}{2}$$
$$\times\left[e^{-(n-m)|\omega|/2}-e^{-(n+m)|\omega|/2}\right] \tag{5.5d}$$

Note also some properties of the matrices which we shall use below:

$$(C^{(n)})^{-1}_{jk} \equiv A^{(n)}_{jk}(\lambda) = \int_{-\infty}^{+\infty} e^{-i\omega\lambda} \, 2\coth\frac{\omega}{2} \frac{\sinh[\min(j,k)\omega/2]}{\sinh(n\omega/2)}$$

$$\times \sinh[(n - \max(j,k))\omega/2] \frac{d\omega}{2\pi} \tag{5.6a}$$

$$A^{(\infty)} = A, \qquad C^{(\infty)} = C \tag{5.6b}$$

$$A_{mk} - A_{mn}A^{-1}_{nn}A_{nk} = A^{(n)}_{mk} \tag{5.6c}$$

6. THE GROUND STATE

We shall show here that the ground state is formed by coupled λs (5.1) and the two order orbital complexes $\mu^{(j)}_2$s (5.2b). In other words, only σ and $\kappa^{(j)}_2$ are nonzero. The other distributions $\rho = \sigma_m = \kappa^{(j)}_n = 0$ $(n \neq 2)$ in the ground state.

The energy of the distribution (5.4a) is a monotonic function of the rapidities. As a consequence, in the ground state all "particles" occupy half-opened intervals:

$$\sigma(\lambda) = 0 \qquad \text{at} \quad \infty < \lambda < Q \tag{6.1}$$

where Q is yet an unknown parameter.

The integral $\int \sigma(\lambda)\, d\lambda$ diverges at $\lambda \to -\infty$. It is a consequence of the initial linear spectrum approximation which leads to an unbounded spectrum. Therefore one should treat all λs as restricted in the interval $(-\mathscr{D}^{(-)}/2\Gamma, \mathscr{D}^{(+)}/2\Gamma)$ where $\mathscr{D}^{(\pm)}$ are edges of the energy band. So one should replace the term which corresponds to the density of states of the host metal [the first term in the left-hand side of Eq. (5.3b)] with the term

$$\eta_0(\lambda) = \frac{2\Gamma}{\pi} \theta\left(\frac{\mathscr{D}^{(+)}}{2\Gamma} - \lambda\right) \theta\left(\lambda + \frac{\mathscr{D}^{(-)}}{2\Gamma}\right) \tag{6.2}$$

The lower edge $\mathscr{D}^{(-)}$ is related to the total number of particles: $\mathscr{D}^{(-)}$ is related to the total number of particles: $\mathscr{D}^{(-)} = \pi N/L$. Integrating Eq. (5.3a) we find

$$\int_Q^{\infty} \tilde{\sigma}(\lambda)\, d\lambda = \mathscr{D}^{(+)}/\pi \tag{6.3}$$

Therefore in the physical ground state the spectral Eqs. (5.3) are

$$\eta_0(\lambda) + \frac{1}{L} a_2(\lambda - \varepsilon_d/2\Gamma) = \tilde{\sigma}(\lambda) + (1 + a_2) * \sigma(\lambda) - s * A_{22} * \kappa_2^{(1)} \quad (6.4)$$

$$\delta_{j1} s * \sigma(\lambda) = C^{jk} * \kappa_2^{(k)}(\lambda) \quad (6.5)$$

To find the parameter Q one should find the host part of $\tilde{\sigma}$ from Eqs. (6.4), (6.5) and satisfy the condition (6.3):

$$\tilde{\sigma}(\lambda) + \sigma(\lambda) + \int_{-\mathscr{D}^{(-)}/2\Gamma}^{Q} \mathscr{F}(\lambda - \lambda')\, \sigma(\lambda')\, d\lambda'$$

$$= \eta_0(\lambda) + \frac{1}{L} a_2(\lambda - \varepsilon_d/2\Gamma) \quad (6.6)$$

where

$$\mathscr{F}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega\lambda - |\omega|} \frac{\sinh[(n-2)\omega/2]}{\sinh[n\omega/2]} \quad (6.7)$$

From Eq. (6.6) and (6.3) we find

$$Q = -\frac{(n-2)}{2\pi} \ln \mathscr{D}^{(+)}/2\Gamma + O(1) \quad (6.8)$$

Below we use the renormalized level

$$\varepsilon_d^* = \varepsilon_d - 2\Gamma Q \quad (6.9)$$

One can check that the ground state described by Eqs. (6.4), (6.5) is an orbital singlet.

To prove the hypothesis of the ground state and to find the physical properties it is convenient to rewrite the spectral equations in terms of the excitations under the physical ground state. Then we shall use $\tilde{\sigma}$, $\tilde{\kappa}_m^{(j)}$ instead of σ and $\kappa_m^{(j)}$.

Inverting the kernels in Eq. (5.3) and substituting $\kappa_2^{(1)}$ into (5.3a,b) we obtain

$$\kappa_m^{(j)} = A_{j1}^{(n)} * s * (\delta_{m1}\rho + \delta_{m2}\sigma) - C_{mp} * A_{jk}^{(n)} * \tilde{\kappa}_p^{(k)} \quad (6.10a)$$

$$\frac{2\Gamma}{\pi} + \frac{1}{L} a_2(\lambda - \varepsilon_d/2\Gamma) = \tilde{\sigma}(\lambda) + \mathscr{F} * [\sigma(\lambda) + s * \rho(\lambda)]$$

$$+ s * A_{1k}^{(n)} * \tilde{\kappa}_2^{(k)}(\lambda) \quad (6.10b)$$

$$\frac{\Gamma}{\pi}+\frac{1}{L}a_1(\lambda-\varepsilon_d/2\Gamma)=\tilde{\rho}(\lambda)+\mathscr{F}*[s*\sigma(\lambda)+(1+a_2)^{-1}*\rho(\lambda)]$$
$$+a_m*\sigma_m(\lambda)+sA_{1k}^{(n)}*\tilde{\kappa}_1^{(k)}(\lambda) \tag{6.10c}$$

Furthermore, excluding σ from (6.10b,c) and employing (5.6) we have

$$s*\tilde{\sigma}(\lambda)+\frac{1}{L}s(\lambda-\varepsilon_d/2\Gamma)=\tilde{\rho}(\lambda)+A_{nn}^{-1}*\rho(\lambda)+a_m*\sigma_m(\lambda)$$
$$+sA_{1k}^{(n)}\tilde{\kappa}_1^{(k)}-s^2A_{1k}^{(n)}\tilde{\kappa}_2^{(k)} \tag{6.11}$$

Now consider the new monotonic function $\varepsilon(\lambda)=\varepsilon^{(+)}(\lambda)+\varepsilon^{(-)}(\lambda)$ where $\varepsilon^+>0$ and $\varepsilon^-<0$ are positive and negative parts of this function. Let $\varepsilon(Q)=0$ and $\varepsilon^{(\pm)}(\lambda)$ satisfy the equation

$$\varepsilon^{(+)}(\lambda)+\mathscr{F}*\varepsilon^{(-)}(\lambda)=4\Gamma\lambda \tag{6.12}$$

After the simple algebra one can rewrite the energy of the state in terms of excitations only:

$$E=E^{(0)}+\int_Q^\infty \varepsilon^{(+)}(\lambda)\,\sigma(\lambda)\,d\lambda-\int_{-\infty}^{Q}\varepsilon^{(-)}(\lambda)\,\tilde{\sigma}(\lambda)\,d\lambda$$
$$-s*A_{p1}^{(n)}*\varepsilon^{(-)}*\tilde{\kappa}_2^{(p)}+s*\varepsilon^{(+)}*\rho \tag{6.13}$$

where $E^{(0)}$ is the ground state energy.

Equations (6.13) prove the hypothesis of the ground state structure: all terms in (6.13) are positive. Conclusion: the energy minimum is achieved at

$$\tilde{\sigma}(\lambda<Q)=\rho(\lambda)=\tilde{\kappa}_m^{(j)}(\lambda)=0 \tag{6.14}$$

It follows from Equations (6.10) that

$$\sigma_m(\lambda)=0,\qquad \kappa_m^{(j)}(\lambda)=0\qquad (m\neq 2)$$

Note that the auxillary function $\varepsilon(\lambda)$ is expressed in terms of the ground state distributions σ_0, $\tilde{\sigma}_0$ [Eq. (6.6)]

$$\sigma_0=\frac{1}{2\pi}\frac{d\varepsilon^{(-)}}{d\lambda},\qquad \tilde{\sigma}_0(\lambda)=\frac{1}{2\pi}\frac{d\varepsilon^{(+)}}{d\lambda}(\lambda)$$

7. KONDO LIMIT

Now we can derive the spectral equations in the Kondo limit $Q\gg 1$ when $\Gamma\gg T_K\sim\Gamma\exp(\varepsilon_d^*/2\Gamma)$. Quite similar to Ref. 16 we should take $\tilde{\sigma}(\lambda)$ in

(6.10) in the physical ground state, given by Eq. (6.6) and use only the leading asymptotic term

$$\int_Q^\infty s(\lambda-\lambda')\,\tilde{\sigma}_0(\lambda')\,d\lambda' \simeq e^{\pi\lambda}\int_Q^\infty e^{-\pi\lambda'}\tilde{\sigma}_0(\lambda')\,d\lambda' \equiv Ae^{\pi\lambda} \tag{7.1}$$

The first term in (6.13) describes charge density excitations which in the Kondo limit is decoupled from the magnetic excitations. The second term in this limit becomes $2Ae^{\pi\lambda}\rho(\lambda)\,d\lambda$. With the same accuracy $\exp(\varepsilon_d^*/2\Gamma) \ll 1$ we should also consider $\kappa_2^{(j)}$ to be in the ground state, i.e., $\tilde{\kappa}_2^{(j)} = 0$ and $\kappa_2^{(j)}$ is given by (6.5).

Then, according to (6.13) and (6.10a) $\tilde{\kappa}_2^{(j)}$ together with $\kappa_m^{(j)}$ $(m > 2)$ excitations are decoupled from the impurity. From (6.10a) we get that $\kappa_1^{(j)}$ are decoupled, in this case from $\kappa_m^{(j)}$. The equation for $\tilde{\kappa}_1^{(j)}$

$$\kappa_1^{(j)} + A_{jk}^{(n)} * \tilde{\kappa}_1^{(k)} = A_{j1}^{(n)} s * \rho \qquad (j = 1,\ldots, n-1) \tag{7.2a}$$

together with equations where all arguments are shifted to $\lambda \to \lambda - 1/\pi \ln A$:

$$\begin{aligned}\tilde{\rho}(\lambda) + A_{nn}^{-1} * \rho(\lambda) + a_m * \sigma_m(\lambda) + sA_{1m}^{(n)}\tilde{\kappa}_1^{(m)} \\ = e^{-\pi\lambda} + (1/L)\,s(\lambda + 1/J)\end{aligned} \tag{7.2b}$$

$$a_m \rho = \tilde{\sigma}_m + A_{mk} * \sigma_k, \qquad \frac{1}{J} = -\frac{\varepsilon_d}{2\Gamma} - \frac{1}{\pi}\ln A \tag{7.2c}$$

are the universal spectral equations for the physical excitations in the Kondo limit.

In the ground state with the given spin projection $\rho \neq 0$, $\tilde{\kappa}_1^{(j)} = 0$, $\sigma_l = 0$. Then the distributions ρ and $\tilde{\rho}$ satisfy the equation

$$\tilde{\rho}(\lambda) + A_{nn}^{-1} * \rho(\lambda) = e^{-\pi\lambda} + \frac{1}{L}\,s(\lambda + 1/J) \tag{7.3}$$

where owing to monotonic dependence of the energy density on the rapidity

$$\rho(\lambda) = 0 \quad \text{at } \lambda > -B, \qquad \tilde{\rho}(\lambda) = 0 \quad \text{at } \lambda < -B$$

The parameter B is defined by the condition

$$S^z = \frac{1}{2}\int_{-\infty}^{-B} \rho(\lambda)\,d\lambda \tag{7.4}$$

Below we shell investigate Eqs. (7.3), (7.4) to find the impurity magnetization (2.11).

8. THE EQUIVALENT MODEL AND GENERALIZATION FOR ARBITRARY IMPURITY SPIN

As has been mentioned in Section 3 the Anderson model (3.2) yields an integrable regularized version of the exchange model (1.1) only for $n = 2S$. Unfortunately, generalization of the integrable Anderson model for arbitrary impurity spin is unknown. Therefore it is important to find the proper regularization of the exchange Hamiltonian.

Here we will give such regularization.

First introduce the notations. Let $v_p^{(1)}$ $(p = 1,..., n)$ be a linear space of the $\frac{1}{2}$-spinor particle in the p-channel state, $V^{(2S)}$ is the space of the S-spinor impurity. The Hamiltonian can be considered as a linear operator acting in the tensor production of these spaces:

$$V_{\mathscr{H}} = \prod_{\alpha=1}^{N} \otimes (v_1^{(1)} \otimes \cdots \otimes v_n^{(1)})_\alpha \, V^{(2S)}$$

The space $v_1^{(1)} \otimes \cdots \otimes v_n^{(1)}$ can be decomposed into a sum of irreducible subspaces $V^{(1)}$ with the highest weights $\Lambda = [m_1, m_2,..., m_n]$ $(m_1 + \cdots + m_n = n)$

$$v_1^{(1)} \otimes \cdots \otimes v_n^{(1)} = \sum_{\Lambda} V^{(\Lambda)}$$

Let $V^{(n)}$ be the space with $\Lambda = [n]$, i.e., the space of symmetric n-rank spinors.

We claim that the proper regularization of the exchange Hamiltonian (1.1) is to consider it both in the whole space and in the subspace

$$V_{\mathscr{H}}^{\text{reg}} = \prod_{\alpha=1}^{N} \otimes (V^{(n)})_\alpha \otimes V^{(2S)}$$

In other words only electron states which are symmetric n-rank spinors should be taken into account as the low-energy states. The other states give no contribution to the low-energy properties of the impurity.

To give proof first construct an integrable exchange model describing the scattering of n-rank symmetric spinors $C\{a_1,..., a_n\}$ by the spin S impurity. Let c_a $(a = 1,..., n)$ be the $n/2$ spin electron operator. Then a the general form of the exchange Hamiltonian is

$$\mathscr{H} = \sum_{k,a} (k - k_F) c_{ka}^+ c_{ka} + J \sum_{\substack{a,b' \\ k,k'}} c_{ka}^+ P(\mathbf{S} \cdot \mathscr{L}, J)_{ab} c_{k'b} \tag{8.1}$$

where $\mathscr{L} = (\mathscr{L}_x, \mathscr{L}_y, \mathscr{L}_z)$ is the operator of the spin $n/2$, and P is still an arbitrary polynomial of the order $\min(2S, n)$.

The electron-impurity S matrix is

$$S = \exp[iJP(\mathbf{S} \cdot \mathscr{L})] \tag{8.2}$$

For the model to be integrable this matrix should belong to the family $\mathscr{L}(u)$; $S = \mathscr{L}$ $(u = 1/J)$ satisfy the factorizable conditions:

$$\mathscr{L}_{12}(u)\,\mathscr{L}_{23}(u+u')\,\mathscr{L}_{13}(u') = \mathscr{L}_{13}(u')\,\mathscr{L}_{23}(u+u')\,\mathscr{L}_{12}(u) \tag{8.3}$$

Such family is known and is obtained if to apply the bound state fusion procedure[17,18] of tensoring the fundamental S matrices (the both spins are $\frac{1}{2}$) (4.2a)

$$\mathscr{L}_{AB}(u) = P_A^+ \prod_{k,l=1}^{n} S_{a_i b_i}(u + i(n+1-k-l))\, P_B^+ \tag{8.4}$$

where $A = \{a_1, \ldots, a_n\}$, P_A^+ is symmetrizator, i.e., a projector to the irreducible subspace $V^{(n)}$. To prove that (8.4) satisfies (8.3) it is sufficient that

$$S_{ab}(i/2) = P^+_{\{ab\}}$$

The explicit form of the polynomial is

$$P(x, \mathscr{E}) = \sum_{l=|n/2-S|}^{n/2+S} \prod_{k=0}^{l} \frac{1-ikJ}{1+ikJ} \prod_{\substack{p=0 \\ p\neq l}}^{\min(n/2,S)} \frac{x - x_p}{x_l - x_p} \tag{8.5}$$

$$x_p = (1/2)\,p(p+1) - (1/2)\,S(S+1) - (n/4)(n/2+1)$$

The Hamiltonian (8.1) is the projection of the Hamiltonian (1.1) to the subspace $V_{\mathscr{L}}$.

To prove this statement and the validity of the regularization we will solve the model below and show that at $n = 2S$ the spectral equations of this model and the Anderson model (3.2) are identical.

9. THE BETHE-ANSATZ SOLUTION OF THE EQUIVALENT MODEL

Application of the Bethe method to the model (8.1) with the approximations cited in Section 3 does not encounter a difficulty. Imposition of the periodic boundary conditions leads to the eigenvalue problem

$$e^{ikjL} = \mathscr{L}_{jj+1} \cdots \mathscr{L}_{jN} \mathscr{L}_{j1} \cdots \mathscr{L}_{jj-1} \tag{9.1}$$

This problem is completely solved.

This fact also helps find a solution of the eigenvalue problem (9.1). It is completely solved when the initial 1-rank problem for the operator

$$T_i(k_i) = \prod_{p=1}^{N} S_{ip}(k_i - k_p) \tag{9.2}$$

is solved for arbitrary ks. Actually if (8.2) is used it is clear that

$$\prod_{k=1}^{n} t_i\left(iJ\left(\frac{n+1}{2} - k\right)\right) = e^{ik_iL} \tag{9.3}$$

where t_i is the eigenvalue of T_i.

Using the explicit form of t_i (4.8) we come to the Bethe-Ansatz equations:

$$\exp(ik_jL) = \prod_{\alpha=1}^{M} e_n(\Lambda_\alpha) \tag{9.4a}$$

$$e_n^N(\Lambda_\alpha)\, e_{2S}(\Lambda_\alpha + 1/J) = \prod_{\beta=1}^{M} e_2(\Lambda_\alpha - \Lambda_\beta) \tag{9.4b}$$

10. THE SPECTRAL EQUATIONS FOR THE EQUIVALENT MODEL

In the thermodynamic limit all solution are grouped into complexes of the order

$$\Lambda_\alpha^{(m,k)} = \Lambda_\alpha^{(m)} + i\left(\frac{m+1}{2} - k\right), \qquad (k = 1,..., m) \tag{10.1}$$

Let f_m and $\tilde{f}_m$ be distributions of the $\Lambda^{(m)}$s and there holes. Then Eqs. (9.4) in the continuous limit are

$$\tilde{f}_m + A_{mk} * f_k = A_{mn} * s(\lambda) + \frac{1}{L} A_{m,2S} * s(\lambda + 1/J) \tag{10.2}$$

The spin projection is

$$S^z = \frac{N}{2} - N \sum_{m=1}^{\infty} m \int f_m(\lambda)\, d\lambda \tag{10.3}$$

The ground state of the model formed by n complexes is

$$f_m = \delta_{nm} f_n \tag{10.4}$$

Therefore to obtain the spectral equations for the physical excitations exclude the ground state configuration. Using the properties (5.6) of the matrix A_{mk}

$$\begin{aligned} f_n + A_{nn}^{-1} * \tilde{f}_n + a_{m-n} * f_m + A_{m,n-1}^{(n)} s * f_m \\ = e^{+\pi\lambda} + (1/N) A_{n,2S} * A_{nn}^{-1} * s(\lambda + 1/J) \end{aligned} \tag{10.5}$$

$$\tilde{f}_{m+n-1} + A_{mk} * f_{k+n-1} = a_m \tilde{f}_n, \qquad m, k = 1, 2, \ldots$$

$$\tilde{f}_m + A_{mk}^{(n)} * f_k = A_{m,n-1} * s * \tilde{f}_n, \qquad m, k = 1, \ldots, n-1$$

The energy and the spin projection expressed in terms of $\tilde{f}_n$ and f_m $(m \neq n)$ are

$$S^z = \theta(s - n/2)(s - n/2) + \frac{N}{2} \int \tilde{f}_n(\lambda)\, d\lambda - \sum_{m=n+1}^{\infty} (m - n)$$

$$\times N \int f_m(\lambda)\, d\lambda \tag{10.6}$$

$$E = \text{const} + N \int s(\lambda) \tilde{f}_n(\lambda)\, d\lambda \tag{10.7}$$

The last expression proves statement (10.4). In the physical ground state where $S^z = 0$ there no holes in the distribution of n complexes.

Now let us compare the spectral equations (10.5) and (7.2) for the models (8.1), (1.1). Their identity is proved by the relation.

It proves that the models (1.1) and (8.1) are identical at least at $n = 2S$. Our hypothesis is that it is also true for arbitrary n and S.

11. THE MAGNETIZATION OF THE IMPURITY AS A FUNCTION OF MAGNETIC FIELD

Consider the Eqs. (10.5) for arbitrary n and S in the ground state with a given spin projection. In this state all f_m (except f_n) $= 0$, $f_n(\lambda > -B) \neq 0$. Decompose the distribution f_n into the host and impurity parts:

$$\tilde{f}_n(\lambda) = f_h(\lambda) + \frac{1}{L} f_{\text{imp}}(\lambda) \tag{11.1}$$

Shifting the arguments in f yields the universal equations:

$$\frac{1}{2} \int_{-\infty}^{0} f_h(\lambda)\, d\lambda = 1 \tag{11.2a}$$

$$M_{\text{imp}} = \frac{1}{2}\int_{-\infty}^{0} f_{\text{imp}}(\lambda)\, d\lambda \tag{11.2b}$$

$$f_h(\lambda) + \int_{-\infty}^{0} K(\lambda - \lambda') f_h(\lambda')\, d\lambda' = e^{\pi\lambda}, \qquad K = -1 + A_{nn}^{-1} \tag{11.3a}$$

$$f_{\text{imp}}(\lambda) + \int_{-\infty}^{0} K(\lambda - \lambda') f_{\text{imp}}(\lambda')\, d\lambda' = A_{nn}^{-1} * A_{n,2S} * s\left(\lambda + \frac{1}{\pi}\ln H/T_H\right) \tag{11.3b}$$

Equations (11.3) can be solved by the Wiener–Hopf method similarly to Ref. 19. Their solutions are

$$f_h(\lambda) = \frac{G^{(+)}(i\pi)}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega\lambda}\frac{G^{(-)}(\omega)}{i\omega + \pi}\, d\omega \tag{11.4a}$$

$$f_{\text{imp}}(\lambda) = \frac{i}{(2\pi)^2}\int_{-\infty}^{+\infty} e^{-i\omega\lambda} G^{(-)}(\omega)\, d\omega \int_{-\infty}^{+\infty} \frac{d\omega'\, G^{(+)}(\omega')}{\omega - \omega' - i0} \times \frac{\exp(-|n-2S|\,|\omega'|/2) - \exp[-(n+2S)\,|\omega'|/2]}{(1 - e^{-n|\omega'|})\, 2\cosh(\omega'/2)} \times \exp(-i\omega'/\pi \ln H/T_H) \tag{11.4b}$$

where

$$G^{(-)}(\omega) = G^{(+)}(-\omega) = \frac{(2\pi n)^{1/2}\left(\dfrac{i\omega n + 0}{2\pi e}\right)^{i\omega n/2\pi}\Gamma(1 + i\omega/2\pi)}{\Gamma(1/2 + i\omega/2\pi)\,\Gamma(1 + i\omega n/2\pi)} \tag{11.5}$$

are analytical functions in the lower and upper half-planes, respectively.

From (11.4a) and the condition (11.2a) we find

$$B + 1/J = \ln(H/T_H) \tag{11.6}$$

where the scale

$$T_H/T_K = \frac{2\pi(n/2e)^{n/2}}{\Gamma(n/2)}, \qquad T_K = \frac{2\varepsilon_F}{\pi} e^{-\pi/J}$$

is the Kondo temperature.

The condition (11.2b) together with (11.4b) yield the impurity magnetization the expression of which we have already discussed in Section 2.

12. LARGE-n LIMIT

Let us discuss the behavior of the impurity magnetization at $n \to \infty$. First consider the case $n/2S$ fixed. Then $(1/n)\,|\ln H/T_H|$ (also should be kept fixed) and

$$M_{\text{imp}} = -\frac{in}{4\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{\omega - i0}\,\frac{e^{i2\omega}(i\omega+0)^{x\omega}}{\Gamma(1+i\omega x)}\,\frac{\sinh[(2S/n)\,x\omega]}{\sinh \omega x} \tag{12.1}$$

where we set $x = nz_0$ where z_0 is given by (2.5).

The expression reveals that the Gell-Mann–Low function (2. at $n \to \infty$ has a limit

$$\lim_{n\to\infty} \beta(n, J) = J^2\beta(nJ) \tag{12.2}$$

Therefore the fixed point is the order of $1/n \sim J^*$. Note that it can hardly be treated by the $1/n$ expansion. The expressions (12.1), (12.2) can be treated as a new theory with the two-loop approximation

$$\beta(x) = -x^2 + x^3 \tag{12.3a}$$

$$\frac{1}{x} + \frac{1}{2}\ln x = \ln H/T_H \tag{12.3b}$$

$$M_{\text{imp}} = \sum_{n=0}^{\infty} a_n x^n \tag{12.3c}$$

Next consider the case when $n - 2S \geqslant 0$ is kept fixed including the $n = 2S$ case. In this case $\ln(H/T_H)$ should be kept fixed:

$$M_{\text{imp}}(H) = -\frac{i\sqrt{n}}{8\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{(\omega - i0)^{3/2}} \times \frac{\exp[2i\omega\ln(H/T_H) - \pi(n-2S)\,|\omega|\,]\,\Gamma(1+i\omega)}{\cosh(\pi\omega)\Gamma(1/2+i\omega)} \tag{12.4}$$

At $\ln(H/T_H) \gg 1$

$$M_{\text{imp}}(H) = \frac{1}{\pi}\,[n\ln(H/T_H)]^{1/2}[1 + O(1/\ln(H/T_H))] \tag{12.5}$$

and smoothly depends on n and S.

The difference between the cases $n = 2S$ and $n > 2S$ arises at $H \ll T_H$. There is the Fermi liquid law for $n = 2S$.

$$M_{\text{imp}}(H) = \frac{\sqrt{n}}{4\pi} \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{k!\,(k+\frac{1}{2})^{1/2}} (H/T_H)^{2k+1} \tag{12.6}$$

For $n - 2S > 0$, from (12.4) we have

$$M_{\text{imp}}(H) = \frac{\sqrt{n}\,(n-2S)}{[\ln(T_H/H)]^{1/2}} [1 + O(1/\ln(T_H/H))] \tag{12.7}$$

This region can also be controlled by the perturbation theory. Factually at $z \gg 1/n$, the leading term in (2.3) is the second-loop contribution:

$$dz/d \ln H = -nz^3$$

It means the physical quantities can be expanded in series in $[\ln(H/T_K)]^{-1/2}$.

13. CONCLUSION

The multichannel Kondo problem is an interesting example of the quantum many-body theory which is asymptotically free at high energy scale and has three different kinds of low-energy behavior depending on the parameters n and S. There are two strong coupling limit fixed points of different character at $n < 2S$ and $n = 2S$ and finite coupling fixed point at $n > 2S$ obeying the scaling low-energy power laws.

All these regimes are studied on the basis of the exact solution. The ground state is the spin and orbital singlet, but at $n > 2S$ it has infinite polarizability.

The magnetic susceptibility is

$$\chi \sim \begin{cases} H^{-1+2/n} & \text{at} \quad T = 0,\ H \ll T_K \\ T^{-1+4/n+2} & \text{at} \quad H = 0,\ T \ll T_K \text{ (Ref. 15)} \end{cases} \tag{13.1}$$

According to Eq. (13.1) the relative dimensionality between the magnetic field and temperature differs from unity

$$H \sim T^{\Delta}, \qquad \Delta = n/n + 2$$

It means that in the scaling regime $H,\ T \ll T_K e^{-n}$ any physical quantity has the characteristic behavior

$$f(H, T) = T^{\Delta} f(H/T^{\Delta};\ T_K^{\Delta^{-1}})$$

where Δ_f is the dimension of the quantity.

In the end of the paper we note that now all known models traditionally studied in the theory of dilute magnetic alloys are integrable and solved exactly.

ACKNOWLEDGMENTS

We are grateful to V. A. Fateev and S. V. Pokrovski for valuable discussions.

REFERENCES

1. N. Andrei, K. Furuya, and J. H. Lowenstein, *Rev. Mod. Phys.* **55**:331 (1983).
2. P. B. Wiegmann, in *Quantum Theory of Solid State*, I. M. Lifshitz, ed. (Mir, Moscow, 1982).
3. A. M. Tsvelick and P. B. Wiegmann, *Adv. Phys.* **32**:453 (1983).
4. J. R. Schrieffer, *Appl. Phys.* **38**:1143 (1967).
5. P. Nozieres and A. Blandin, *J. Phys. (Paris)* **41**:193 (1980).
6. A. M. Polyakov and P. B. Wiegmann, *Phys. Lett. B* **141B**:221 (1984).
7. A. M. Tsvelick and P. B. Wiegmann, *Z. Phys.* **54**:201 (1984).
8. A. A. Abricosov and A. A. Migdal, *J. Low Temp. Phys.* **3**:519 (1970).
9. M. Fowler and A. Zawadowski, *Solid State Commun.* **9**:471 (1971).
10. V. A. Fateev and P. B. Wiegmann, *Phys. Lett.* **81A**:179 (1981); *Phys. Rev. Lett.* **46**:356 (1981).
11. P. Nozieres, *J. Low Temp. Phys.* **17**:31 (1974).
12. K. Furuya and J. H. Lowenstein, *Phys. Rev. B* **25**:593 (1982).
13. P. B. Wiegmann, *Phys. Lett.* **80A**:163 (1980).
14. P. A. Schlottman, *Phys. Rev. Lett.* **50**:1697 (1983).
15. P. B. Wiegmann and A. M. Tsvelick, *J. Phys. C* **16**:2281 (1983); A. M. Tsvelick and P. B. Wiegmann, *J. Phys. C* **16**:2321 (1983).
16. A. M. Tsvelick, *J. Phys. C.* **17**, to be published (1984).
17. V. A. Fateev, unpublished.
18. P. P. Kulish, N. Y. Reshetichin, and E. Sklyanin, *Lett. Math. Phys.* **5**:393 (1981).